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MATH 254

Project 2

The origin of many formulas and principles in calculus comes from the trial-and-error of many mathematicians, which is evidently shown in the integrating factor of first-order differential equations. The following tasks go through the process of laying the foundation for the integrating factor.

Task 1: What are the “curve,” “tangent line,” and “particular law” of differential equations in the form $\frac{dy}{dx} = f(x,y)$?

The “curve” is the solution to the differential equation, giving the change over time.

The “tangent line” is the slope of that curve at any given x,y coordinate.

The particular law is the constraints that the equation must follow, or the context of the problem you are solving for.

Task 2: How do you turn a non-homogeneous differential equation into a monic differential equation? Why can we assume p(x) isn’t identically zero? Write P(x) and Q(x) in terms of p(x), q(x), and f(x).

Non-homogenous form: $p(x)\frac{dy}{dx} + q(x)y = f(x)$

Monic form: $\frac{dy}{dx} + P(x)y = Q(x)$

$$p(x)\frac{dy}{dx} + q(x)y = f(x) \quad => \quad \frac{dy}{dx} + \frac{q(x)y}{p(x)} = \frac{f(x)}{p(x)}$$

Showing $P(x)y = \frac{q(x)y}{p(x)}$ and $Q(x) = \frac{f(x)}{p(x)}$

You can assume p(x) isn’t identically zero because it would result in $\frac{dy}{dx}$ becoming zero in the non-homogenous form, and it would make P(x) and Q(x) undefined in the monic form.

Task 3: How did Leibniz go from $\int mpdx + py = 0$ to $mpdx + ydp + pdy = 0$?

Taking the derivative of each term, using the product rule ($(f(x)g(x))^{l} = f^{l}(x)g(x) + f(x)g^{l}(x)$) for py, has the following results:

$\frac{d}{dx}\int mpdx + \frac{d}{dx}py = 0 \ => mpdx + dpy + pdy = 0$

Explain how Leibniz went from $\int \frac{dp}{p} = \int ndx$ to $dp = pndx$.

If you take the derivative of each side, and then multiply each term by p, you will get the result:

$\frac{d}{dx}\int \frac{dp}{p} = \frac{d}{dx}\int ndx$ => $\frac{dp}{p} = ndx$ => $dp = pndx$ *any resulting constants are insignificant*

How did Leibniz combine the first two problems to obtain $mdx + nydx + dy = 0$?
Starting with the first equation, if you divide each term by p, and then substitute dp with $pndx$ from the second equation, you will get the result:
$mpdx + dpy + pdy = 0$ => $mdx + \frac{dpy}{p} + dy = 0$ => $mdx + \frac{pndxy}{p} + dy = 0$
=> $mdx + ndxy + dy = 0$

Why is this the "desired" result?
Leibniz started with a potential solution to the original problem, and this result is ideal because it proves that it is a solution.

Task 4: Turn the implicit solution into an explicit one by solving for y.

The implicit solution can be turned into an explicit one by moving mdx to the other side, dividing each term by dx, moving $\frac{dy}{dx}$ to the other side, and then dividing by n.

$mdx + nydx + dy = 0$ => $nydx + dy = -mdx$ => $ny + \frac{dy}{dx} = -m$ =>

$$ny = -m - \frac{dy}{dx} \Rightarrow y = \frac{-m-\frac{dy}{dx}}{n}$$

Task 5: Consider the equation $\frac{dy}{dx} = \frac{\alpha}{(y-x)}$. Switch the variables x and y, solve for $\frac{dy}{dx}$, and then compare to previous equations (1 and 2). How does this relate to P(x) and Q(x)?

$\frac{dy}{dx} = \frac{\alpha}{(y-x)}$ switching the variables → $\frac{dx}{dy} = \frac{\alpha}{(x-y)}$. From here you can multiply both sides by $(x - y)$, multiply by $\frac{dy}{dx}$, and divide by α.

$\frac{dx}{dy} = \frac{\alpha}{(x-y)}$ => $\frac{dx}{dy}(x - y) = \alpha$ => $x - y = \alpha\frac{dy}{dx}$ => $\frac{dy}{dx} = \frac{1}{\alpha}x - \frac{1}{\alpha}y$

Comparing this to $p(x)\frac{dy}{dx} + q(x)y = f(x)$ and $\frac{dy}{dx} + P(x)y = Q(x)$, we can see that there is a relation. By rewriting $x - y = \alpha\frac{dy}{dx}$ to $\alpha\frac{dy}{dx} + y = x$, we can see that $p(x) = \alpha$, with $q(x)y = y$ and $f(x) = x$. $P(x)y = \frac{q(x)y}{p(x)} = \frac{y}{\alpha}$ and $Q(x) = \frac{f(x)}{p(x)} = \frac{x}{\alpha}$

Task 6: Using $x\frac{dy}{dx} + y = 3x^2$, rewrite in the form Leibniz uses to begin his process, what are m and n, solve for p, verify the solution solves the original differential equation, and solve for y to show it solves the equation.

Leibniz started with a monic equation, so we need to rewrite the equation, and we also need to see the relationship between the monic form we know and the implicit solution used earlier.

****************** Used repeatedly in the rest of this assignment *******************
The monic form we know is $\frac{dy}{dx} + P(x)y = Q(x)$, and we need to manipulate this into the form$mdx + nydx + dy = 0$. By dividing each term by dx and moving everything to one side, we get

$\frac{dy}{dx} + P(x)y = Q(x)$ => $\frac{dy}{dx} + P(x)y + (-Q(x)) = 0$ =>

$dy + P(x)ydx + (-Q(x))dx = 0$, where $P(x) = n$ and $Q(x) =- m$ in the implicit solution.

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Now we should rewrite $x\frac{dy}{dx} + y = 3x^2$ as $\frac{dy}{dx} + \frac{1}{x}y = 3x$ after dividing by x. We can see from the general monic form that $P(x) = \frac{1}{x}$ and $Q(x) = 3x$, and we just showed that this means $n = \frac{1}{x}$ and $m =- 3x$.

Leibniz also gives us $\frac{dp}{p} = \frac{dx}{x}$, which we can use to solve for p.

$\frac{dp}{p} = \frac{dx}{x}$ => $\int\frac{dp}{p} = \int\frac{dx}{x}$ => $ln(p) = ln(x)$ => $p = x$

We can substitute those values into $\int mpdx + py = 0$ to see if it is a solution to the original differential equation.

$\int mpdx + py = 0$ => $\int - 3x(x)dx + xy = 0$ => $- x^3 + xy = 0$

By taking the derivative of this, we can show that it is our original equation.

$- x^3 + xy = 0$ => $\frac{d}{dx}(- x^3 + xy) = 0$ => $- 3x^2dx + xdy + ydx = 0$ =>

$xdy + ydx = 3x^2dx$ => $dy + \frac{ydx}{x} = 3xdx$ => $\frac{dy}{dx} + \frac{1}{x}y = 3x$

This does equal the original equation. We can also solve the solution for y and prove it is a solution.

$- x^3 + xy = 0 \;\; => \;\; xy = x^3 \;\; => \;\; y = x^2 \;\; => \;\; y^l = 2x$

$\frac{dy}{dx} + \frac{1}{x}y = 3x \;\; => \;\; 2x + \frac{1}{x}x^2 = 3x \;\; => \;\; 3x = 3x$

Task 7: Follow the steps of task 6, solving for $\frac{dy}{dx} - y = xe^x$.

This equation is already in monic form, and we have already proved the relationship between the monic form and the implicit solution, so we can see that $n = - 1$ and $m = - xe^x$.

$\int \frac{dp}{p} = \int n dx \;\; => \;\; ln(p) = - x \;\; => \;\; p = e^{-x}$

We can then substitute m and p into $\int mpdx + py = 0$ to find what should be the solution to the original equation.

$\int mpdx + py = 0 \;\; => \;\; \int - xe^x(e^{-x})dx + (e^{-x})y = 0 \;\; =>$ - $\int xdx + e^{-x}y = 0 \;\; =>$

$- \frac{1}{2}x^2 + e^{-x}y = 0$

Taking the derivative of that: $\frac{d}{dx}(- \frac{1}{2}x^2 + e^{-x}y) = 0 \;\; =>$

$- \frac{1}{2}(2)xdx + (- e^{-x})y + e^{-x}dy = 0 \;\; => \;\; - xdx - e^{-x}dxy + e^{-x}dy = 0 \;\; =>$

$dy - ydx - e^x xdx = 0 \;\; => \;\; \frac{dy}{dx} - y - e^x x = 0 \;\; => \;\; \frac{dy}{dx} - y = xe^x$, which is equal to the original differential equation. We can also solve for y to see if it is equal.

$- \frac{1}{2}x^2 + e^{-x}y = 0 \;\; => \;\; e^{-x}y = \frac{1}{2}x^2 \;\; => \;\; y = \frac{1x^2}{2e^{-x}} = \frac{x^2e^x}{2} \;\; => \;\; y^l = e^x x + \frac{e^x x^2}{2} \;\; =>$

$\frac{dy}{dx} - y = xe^x \;\; => \;\; e^x x + \frac{e^x x^2}{2} - \frac{e^x x^2}{2} = e^x x \;\; => \;\; e^x x = e^x x$

Task 8: Show that μ is the same as p. $\frac{dy}{dx} + P(x)y = Q(x)$ $\quad \mu = e^{\int P(x)dx}$

$\int \frac{dp}{p} = \int n dx$

We proved earlier that $P(x) \ = \ n$, which can be used to help us work through this relationship.

Using $\int \frac{dp}{p} = \int ndx$, and only taking the derivative of one side => $ln(p) \ = \ \int ndx$ =>

$p \ = e^{\int ndx}$

This can be rewritten as $p \ = e^{\int P(x)dx}$, which means $\mu \ = \ p$

In conclusion, I greatly appreciate the fact that I have the ability to take calculus after all the hard stuff has been figured out for me. These tasks showed us the process of finding the integrating factor of first-order differential equations through multiple forms of equations and thinking processes.